International Journal of Computer Engineering in Research Trends
ISSN: 2349-7084/ https://doi.org/10.22362/ijcert.v10i11.921
Volume 10, Issue 11, November 2023

Research Paper

Geometrical aspects of the algebraic number related to quasicrystals

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Received: 08/08/2023, Revised: 23 /09/2023, Accepted: 09/10/2023 Published: 28/11/2023

Abstract: - *A quasicrystal is a mathematical term for an infinite point or tiling space. It possesses several intersecting features, including Delone, relative discreteness, and self-similarities. The model set is the basic form in which the physical quasicrystals are represented. Pisot numbers are used to identify the one-dimensional model sets' adjacent points. The quadratic irrational numbers are related to all one-dimensional model sets that have been experimentally discovered the paper offers mathematical models of quasicrystals with particular attention given to cut and projection sets for the eight folded symmetry and discuss about one dimensional cut and projection set.*

Keywords- *Quasicrystals, cut-and-project set, cut-and-project sequence, Meyer's theory, Delone sets*

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1. Introduction

A quasicrystal is a mathematical term that refers to either an infinite point set or the tiling of a matching space, depending on the situation [7, 18]. Physics is the source of quasicrystals. The name's meaning is frequently summed up as "aperiodic crystal." Since their discovery in 1984 [19], they have been the focus of extensive experimental and theoretical research; for example, see the early literature in [23] and the proceedings from the most recent global conferences on quasicrystals [22]. However, no accepted definition of a quasicrystal exists that would satisfy a mathematician. In physics, the lack of mathematical precision typically poses no problem when considering the precise properties of genuine quasicrystals.

About ten years before the discovery of quasicrystal line materials in physics, Penrose tilings of the plane, also known as quasicrystals, first appeared in mathematics [17]. De Bruijn then built their algebraic theory [4]. It was discovered much later [15, 16, 9] that quasicrystals can be viewed as a specific instance in Y. Meyer's general theory [14]. Meyer's theory and quasicrystals are related in [3], which is the oldest mention we are aware of." is strongly tied to the beginnings of quasicrystals in science and mathematics. That is the irrationality found in the Penrose tilings and in the diffraction spectra, which were used to discover the existence of quasicrystals. Later, more irrationality was rather readily incorporated into the idea of quasicrystals.

With the obvious exception, quasicrystals have a variety of intriguing lattice-like characteristics. Self-similarities, relative discreteness, uniform density, and perhaps other properties all contribute to periodicity. In general, one needs a Delone mathematical model for quasicrystals. Bragg peaks are created by the Fourier transform [3, 5]. One of the mathematical definitions of this condition is satisfied by the so-called cut and project scheme, which uses quasicrystals [1, 8]. The points of a slab that has been cut from a higher dimension's crystallographic lattice are then projected onto an appropriate lower-dimensional subspace. Cut and project quasicrystals or model sets are the resulting sets. The potential of an analogous algebraic description of a model set is implied by the additional requirement that a model set has self-similarity. In this instance, an algebraic module over a ring of integers serves as the stage for a cut and project quasicrystal, which takes advantage of the Galois isomorphism in the related algebraic field. The location, configuration, and dimensions of a confined region known as an acceptance window define a particular model set.

The projection of the lattice Z2 on two straight lines is the simplest basic illustration of a cut and project system. One may demonstrate that the self-similarity factor for a model set R arising from such a cut and project scheme must be a quadratic Pisot number [15]. Furthermore, in the quadratic field Q[v], is a subset of a ring of integers.

Researchers discuss the separations between neighbouring points in the one-dimensional model sets based on quadratic unitary Pisot numbers in this work. The 3-gap theorem, which describes results regarding the distribution of integers n (mod 1), relates to the fact that there are only three types of these distances in a given model set (among many articles on the subject, see for example [20]). However, it is not easy to derive the aforementioned feature for model sets from the 3-gap theorem. This was already discovered for the golden ratio in [11], along with the findings for higher dimensional model sets. The physics of quasicrystals is not particularly interesting in the one-dimensional cut and project sets. Their characteristics might, however, prove valuable in other contexts. For instance, its aperiodicity in the strictest sense can be leveraged to enhance the capabilities of random number generators [6].

The one unambiguous criterion that all scientists have agreed upon is that the set used to model the quasicrystal, or the atom locations in the material, must be a Delone set. This property essentially states that atoms in the quasicrystal should be dispersed "uniformly" over the area that the substance occupies. The set is known as Delone if it satisfies uniform discreetness and relative density property. However, the criterion of the Delone property is insufficient because the positions of the atoms in amorphous matter also constitute a Delone set. Delone sets that model quasicrystals must therefore satisfy additional requirements. There are various ways to quasicrystal definitions, depending on the nature of these extra constraints [24, 25]: Fourier analysis forms the basis of the Bohr-Besicovich concept of a nearly periodic set. The second Patterson set idea is based on a Hof-created mathematical model of X-ray diffraction.

A constraint on the set of interatomic distances is the foundation of Yves Meyer's third notion. It is sophisticated and entirely geometric: There exists a finite set S such that , which distinguishes a Meyer set from a Delone set. In [24], Lagarias demonstrated that a Meyer set can be described similarly to a Delone set, demonstrating that is likewise Delone.

The so-called cut and project sets, often known as C&P sets, are a generic family of sets that are known to have quasicrystalline features. These sets' many subclasses appear to meet each of the three definitions of quasicrystals that were previously mentioned.

The following is how the paper is set up: In Section 2, the method of building quasicrystal models by cut and projection is introduced. In Section 3, a cut-and-project set with 8-fold symmetry is used as an illustration. The paper's final section concentrates on the characteristics of cut-and-project sets in one dimension. Their definition is provided in Section 4.

2. Preliminaries cut-and-project sets (C & P)

The selection of a full rank lattice is the first step in the development of a cut-and-project set (C&P set): Let the vectors be linearly independent over R, the set

$$P=\{\sum_{i=1}^{k} a_i t_i \mid a_i \in Z, i=1,2,...k\}$$

is known as lattice. A lattice is unmistakably a Delone set. Impeccable crystal representation in mathematical model is the set Δ in R^k, which is created from a limited number of shifted copies of the lattice P. The model set Δ is said to be an ideal crystal if $\Delta = P+Q$, Q is called the finite set of translations. An ideal crystal satisfies $\Delta-\Delta \subset \Delta - Q$, and lattice fulfils $P-P\subset P$ hence they are both Meyer sets. Thus, the definition of crystals is extended by the Meyer concept of quasicrystals. However, because the perfect crystal is a periodic set, it is not a useful model for quasicrystal line materials, which have rotational symmetries that are incompatible with periodicity. We will now go through a sizable class of Meyer sets that are not translation-invariant. Let R^k be expressed as the direct sum $A_1 \oplus A_2$ of two subspaces and let P be a full rank lattice in R^k. The projection map's orientation is determined by the other subspace, A_2, while one of the subspaces, let's say A_1, acts as the physical space onto which the lattice P is projected. The inner space of A_2 is so named. The projection map on A_1 along A_2 will be denoted by the number η_1, and similarly for the number η_2. The following diagram illustrates the situation:

$$A_1 \xleftarrow{\eta_1} R^k \cup P \xrightarrow{\eta_1} A_2$$

Additionally, the complete rank lattice must be in general position, which calls for the projection η_1 to be one-to-one when applied to the lattice P and the image of the lattice P under projection η_2 to be a set dense in A_2.

The cut and project set are defined as $\Pi(\psi)=\{\zeta_1(t) \mid t\in P, \zeta_2(t)\in\psi\}$

The bounded set $\psi \in A_2$ is called acceptance window. The properties C&P are as follows:

- $\Pi(\psi)+x \not\subset \Pi(\psi), x\in A_1$ then $\Pi(\psi)$ is not an ideal crystal
- If the acceptance window is not empty the $\Pi(\psi)$ is Mayer set.
- Mayer set is a subset of sum of acceptance window and finite set.

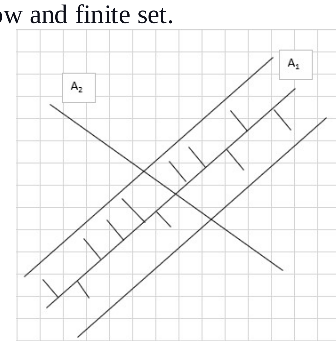


Figure 1 is the one-dimensional cut and projection set with acceptance window as A_2

Finding a mechanism that drives the atoms in quasicrystalline materials to occupy specific locations is the goal of physicists. All physical explanations of crystals are based on the minimal energy thesis. The number of different neighbourhoods of points in the Delone set must be finite if one hopes to at least have a chance of discovering a physical explanation for why a particular

Delone set is a good model for a quasicrystal. This criterion is formalized by the idea of finite local complexity, which states that for each specific radius r, all balls of that radius contain only a finite number of distinct configurations of points up to translation.

Their definition implies that Meyer sets have finite local complexity. A C&P set is therefore guaranteed to have finite local complexity by the condition. Every possible arrangement of points must be present in the modelling set an unlimited number of times for the quasicrystal model to be considered physically plausible. For instance, the requirement that the boundary of the acceptance window have an empty intersection with the projected image of the lattice can provide this property.

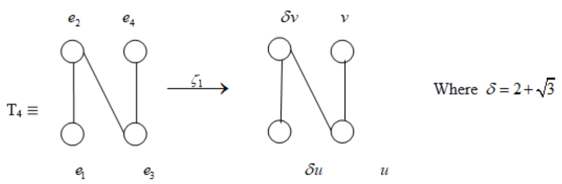
3. Rotational symmetry for cut-and-project set

Consider $P \subset R^4$ be a lattice with unit vectors e_1, e_2, e_3, e_4. The following diagram shows the position of the unit vectors

$$T_4 \equiv e_1 \to e_2 \to e_3 \to e_4$$

The vectors in the diagram that are joined by an edge form a third-angle bend; otherwise, they are orthogonal. These vectors represent the alternative group's root vectors. The produced lattice's invariance under 8-fold rotational symmetry can be confirmed. The least dimension that permits a lattice with such rotational symmetry is 4, so let's mention that.

In this case, the physical space A1 and the interior space A2 are both of dimension 2, making them spanned by two vectors. We can select these two vectors as unit vectors so that u and v create an angle of $4\pi/5$ and conjugate vectors u* and v* are an angle of $2\pi/5$. The definition of the projection is shown in the following diagram, which is uniquely given if it is specified using the four basis vectors e_1, e_2, e_3, e_4.



δ is the irrational number and the root of the equation $x^2 = 2x+1$. Remember that a regular pentagon with side length δ has a diagonal length, which explains how the building of a point set with 8-fold rotational symmetry corresponds to the golden ratio.

Similarly, the projection ζ_2 can be defined by replacing u and v with u* and v* in the diagram. The scalar factor will be the conjugate of the equation $x^2 = 2x+1$ i.e., $\delta' = 2-\sqrt{3}$.

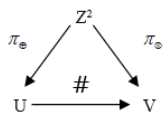
A point of the lattice with integer coordinates (a, b, c, d) and projections ζ_1 & ζ_2 the cut projection set is in the form of

$$\Pi(\psi) = \{(a+b\delta)v+(c+d\delta)u \mid a,b,c,d \in Z, (a+\delta'b)v+(c+\delta'd)u \in \psi\}$$

We must offer the acceptance interval in order to fully define the C&P set. Its choice directly impacts the geometric properties of the C&P set.

4. Cut-and-project set for one dimensional quasicrystal

Most often, projection of lattices with arbitrary dimensions is used as the foundation for discussions of cut and project sets. The one-dimensional subspaces U and V provide the cut and projection in Z2 with projections $\pi_\oplus : Z^2 \to U$ and $\pi_\otimes : Z^2 \to V$ such that $\pi_\oplus$ is one to one mapping in Z2 and $\pi_\otimes$ is dence in Z2 are the properties of projection $\pi_\oplus \& \pi_\otimes$ respectively. The following figure demonstrated this process:



The projections are on Z2 is an additive abelian group and bijective. The mapping $\# : U \to V$ is called # map and -# is called inverse. Let U and V be a linear span of the vectors $\overline{\alpha} = (1, \varepsilon_1) \& \overline{\beta} = (-1, \varepsilon_2)$ where $\varepsilon_1 \neq \varepsilon_2$ which satisfy properties of $\pi_\oplus \& \pi_\otimes$ respectively.

So, any vector $(a,b) \in Z^2$ can be represented as

$$(a,b) = \pi_\oplus(a,b) + \pi_\otimes(a,b)$$

$$(a,b) = \frac{1}{\varepsilon_1+\varepsilon_2}(b+\varepsilon_2 p)\overline{\alpha} + \frac{1}{\varepsilon_1+\varepsilon_2}(b-\varepsilon_1 p)\overline{\beta}$$

If the common factor $1/\varepsilon_1+\varepsilon_2$ is neglected then there will be two abelian groups such as

$$Z[\varepsilon_1] = \{a+b\varepsilon_1 \mid a,b \in Z\} \quad \text{and}$$

$$Z[\varepsilon_2] = \{a+b\varepsilon_2 \mid a,b \in Z\}$$

The mapping

$\# : Z[\varepsilon_1] \to Z[\varepsilon_2]$ given by $\#(a+b\varepsilon_1) = a-b\varepsilon_1$

As # is a linear transformation and also one – one and onto so $Z[\varepsilon_1] \cong Z[\varepsilon_2]$ i.e. $Z[\varepsilon_1]$ is isomorphic $Z[\varepsilon_2]$.

The following definition of a one-dimensional cut-and-project set can be made using this agreement.

Definition: A one dimensional cut and project set is defined as the if irrational numbers $\varepsilon_1, \varepsilon_2 (\varepsilon_1 \neq \varepsilon_2)$ and let the bounded interval ψ (is called as acceptance window) then the set will be of the form

$$\Pi(\psi) = \{a + b\varepsilon_1 \mid a, b \in Z, a + b\varepsilon_2 \in \psi\} = \{t \in Z[\varepsilon_1] \mid t^* \in \psi\}$$

5. Conclusion

The characteristics of one-dimensional models, whether the geometric or combinatorial elements of these structures, are best known in the theory of mathematical quasicrystals. Since one-dimensional models are contained in higher-dimensional ones, this knowledge can also be used to examine such models. In fact, there are infinitely many of them and their ordering is a one-dimensional cut-and-project sequence for any straight line having at least two points from a higher-dimensional cut-and-project set. This technique can also be used to determine the separation between two adjacent points.

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